Lumped-Element Modeling with Equivalent Circuits

Content for these notes taken from:

[1] S.D. Senturia, *Microsystems Design*. USA: Kluwer Academic Publishers, 2001.

[2] Joel Voldman, course materials for 6.777J / 2.372J Design and Fabrication of Microelectromechanical Devices, Spring 2007. MIT OpenCourseWare (http://ocw.mit.edu/), Massachusetts Institute of Technology

Generating Equivalent Circuits [2]

It is possible to go “directly”

- However, it can be hard with the e → V analogy

It is easier to perform the analysis with circuit duals

We can use the convenience of the f → V convention (“indirect”) and then switch to the e → V convention:

- Force is a current source*
- Each displacement variable is a node
- Masses are connected between nodes and ground
- Other elements are connected as shown in the diagram

*Direction of the current source dictates physical motion, but this is “symmetric” in the sense that the same force applied in one direction produces the same behavior as the same force applied in the other direction, push v. pull; it’s just that the displacement is opposite in sign. General rule of thumb, a pull is a positive current, a push is a negative current, but you can always flip the sign at the very end if your results don’t match your expectations

e → V Convention versus f → V Convention [1]

e → V Convention (Direct Analogy):

Mechanical		Electrical		Conversion
Spring	$F = kx \Rightarrow x = \frac{1}{k}F \Rightarrow \dot{x} = \frac{1}{k}\frac{dF}{dt}$	**Capacitor**	$I = C\frac{dV}{dt}$	$k = \frac{1}{C}$
Damper/Dash-Pot	$F = bv = b\dot{x}$	**Resistor**	$V = IR$	$b = R$
Mass	$F = ma \Rightarrow F = m\ddot{x}$	**Inductor**	$V = L\frac{dI}{dt}$	$m = L$

Comparing the expressions in the two domains, we see that in the e → V convention,

- **Force** (effort, e)→ **Voltage** (V): e → V (This is why we refer to this approach as the e→ V convention)
- **Velocity** (flow, f) → **Current** (I)

f → V Convention (Indirect/Mobility Analogy):

Mechanical		Electrical		Conversion
Spring	$F = kx \Rightarrow F = k\int \dot{x}dt$	**Inductor**	$V = L\frac{dI}{dt} \Rightarrow I = \frac{1}{L}\int Vdt$	$k = \frac{1}{L}$
Damper/Dash-Pot	$F = bv = b\dot{x}$	**Resistor**	$V = IR \Rightarrow I = \frac{1}{R}V$	$b = \frac{1}{R}$
Mass	$F = ma \Rightarrow F = m\ddot{x}$	**Capacitor**	$I = C\frac{dV}{dt}$	$m = C$

Comparing the expressions in the two domains, we see that in the f → V convention,

- **Force** (effort, e) → **Current**
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<u>Example: Approach 1 (Complete f → V Analysis)</u>

Find the equivalent electrical circuit of the following mechanical system:

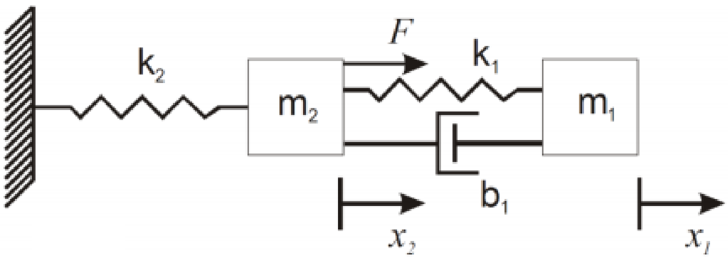

Figure 1: Mechanical System

With the complete f → V analysis, we replace the force with a current source and each of the mechanical elements with their electrical equivalents.

NOTE: Because we are using the f → V convention (indirect/mobility analogy), we need to replace the components as follows, springs with inductors, damping elements/dash pots with resistors, and masses with capacitors according to the corresponding convention):

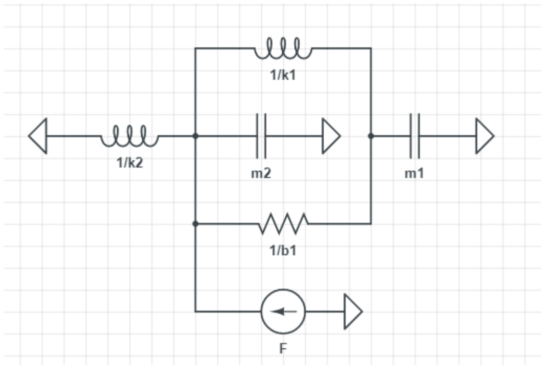

Figure 2: Mechanical to electrical circuit conversion by f → V convention

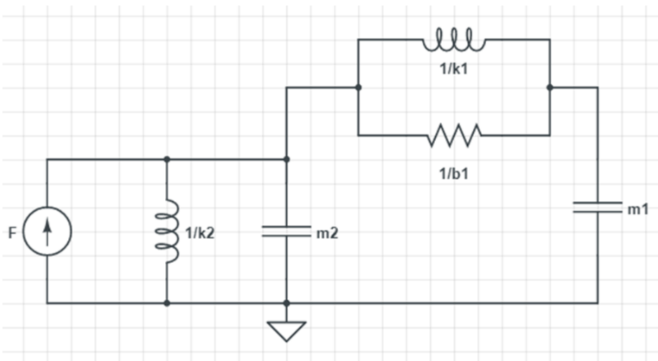

Figure 3: Rearrangement of electrical circuit

The next step is to convert this circuit to its dual, which will put it in the e → V form that we are used to working with. As a quick review, when converting a circuit to its dual, current sources are replaced by voltage sources, inductors become capacitors, resistors become conductances, and vice versa. The value of the components does not change because we need the impedances in the original and dual circuit to be algebraic inverses. (e.g., an inductor with inductance L has impedance $Z = j\omega L$. In the dual, this element becomes a capacitor with capacitance C, so it has impedance $Z' = \frac{1}{j\omega L}$)

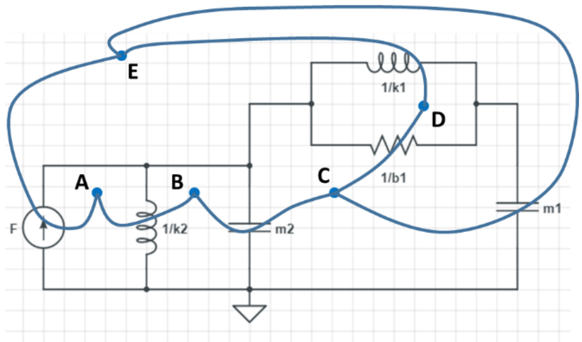

Figure 4: Conversion procedure to circuit dual

<u>Procedure:</u> Draw a point/node (e.g., A, B, C, D) inside each loop, as well as one on the outside (E). Next, draw lines connecting points/nodes: each component that you pass through is converted to its dual and appears between these two nodes. For example, between A and B, you have an inductor, which in its dual becomes a capacitor. In the case of C, there is a connection to D through a resistor/conductance and one to E through the capacitor/inductor.

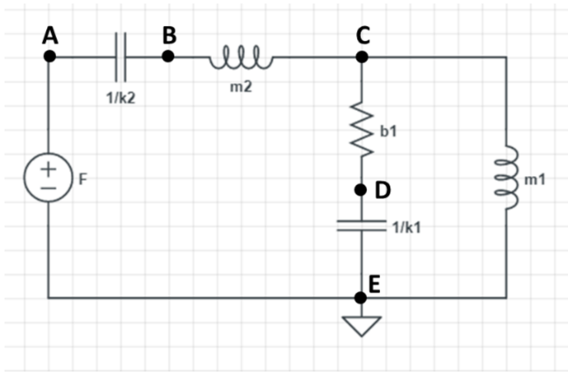

Figure 5: Dual circuit, e → V convention

In Figure 5 we have simply swapped the parallel branches, which does not affect the electrical performance/characteristics. We can also swap elements that are placed in series without any effect on the circuit.

NOTE: For the resistor, **b1**, which represents the damping/dash-pot element, when we convert our f → V circuit to its dual, we see that the impedance changes from $\frac{1}{b_1}$ to b_1. This is because the dual of a resistor is a conductor with a conductance equal to the original resistance (i.e., $G = \frac{1}{b_1}$). However, for consistency, we convert this conductor back to a resistor, which is why we see a resistor of resistance b_1 in the final circuit.

<u>Hybrid Approach [2]</u>

You may have noticed that it is a bit cumbersome to remember the mechanical-electrical parameter equivalents for both the f → V and e → V conversions when we end up using the latter in the end. In other words, it would be nice to convert mechanical equivalents directly to their electrical equivalents according to the rules of the e → V convention: **springs** ⟷ **capacitors** ($k = \frac{1}{C}$), **dampers/dash-pots** ⟷ **resistors** ($b = R$), and **masses** ⟷ **inductors** ($m = L$).

One solution is to implement a hybrid approach:

1. Replace all forces with an equivalent current source (f → V approach)
2. Connect masses between nodes and ground
   Connect the rest of the elements between nodes as in the original schematic (if one terminal of a component is fixed, set that terminal to ground)
3. <u>**Do not**</u> make any conversions between mechanical and electrical elements
4. Redraw the circuit if necessary and determine the "dual circuit".
   - Because we still have mechanical elements, we are not really finding the inverse impedances of our elements (i.e., duals), but rather determining the electrical dual of our mechanical elements
   - This approach produces the same circuit without having to remember the f → V equivalents between mechanical and electrical components

Example: Approach 2 (Hybrid f → V Analysis)

We validate this second approach by analyzing the same circuit from before:

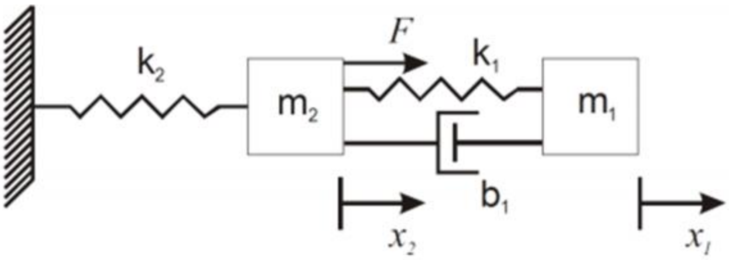

Figure 6: Original mechanical circuit

We replace the force with a current source and define the connectivity of mechanical elements in terms of nodes,

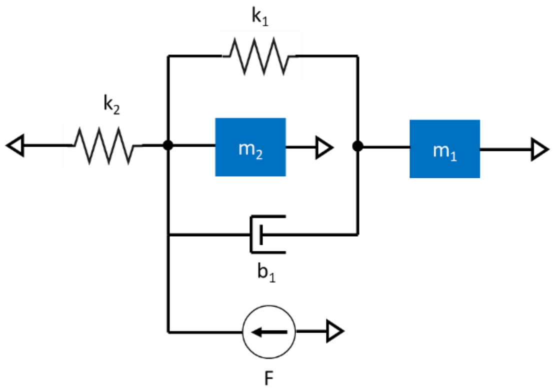

Figure 7: Replace force with a current source and create nodes

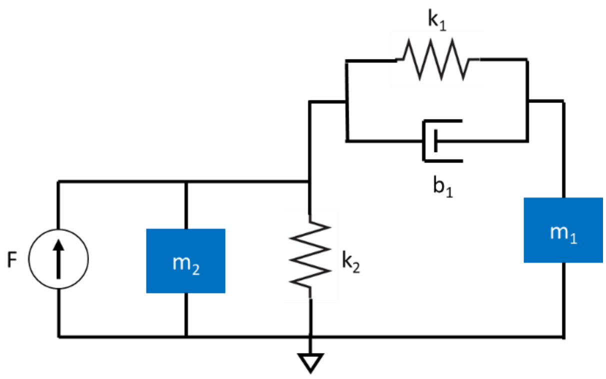

Figure 8: Rearranged mechanical circuit with force represented as a current source

Now, we find the circuit dual in terms of mechanical-electric elements according to the e → V convention, rather than between impedances and admittances. The one exception is the current source, which remains in the electrical domain and becomes a voltage source.

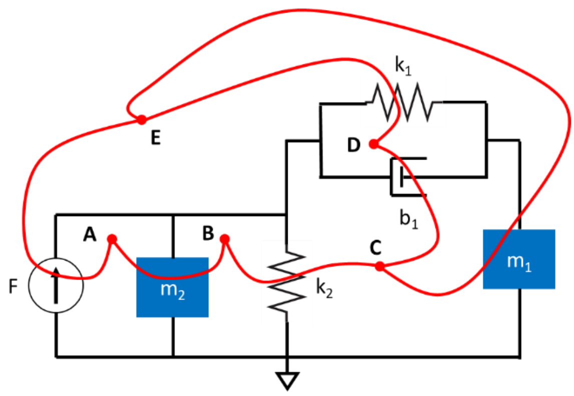

Figure 9: Conversion procedure to circuit dual

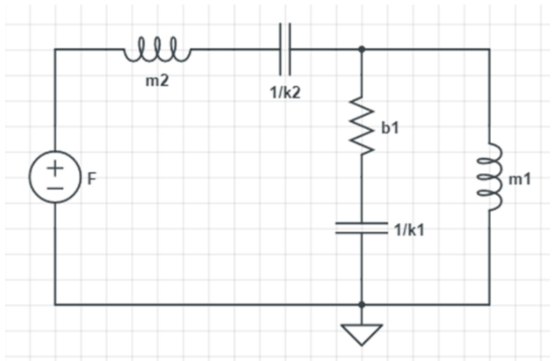

Figure 10: Dual circuit, e → V convention